HAZARD MEASURE AND MEAN RESIDUAL LIFE ORDERING: A UNIFIED APPROACH

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Hazard Measure and Mean Residual Life Ordering: A Unified Approach

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Abstract

The hazard rate ordering is applied frequently in reliability to compare two probability distributions on R_+ such that they are both absolutely continuous (w.r.t. Lebesgue measure) or both purely discrete (concentrated on the set of non-negative integers) via their hazard rates. Kotz and Shanbhag (1980) extended the concept of hazard rate introducing new concept of hazard measure, applicable to any arbitrary distribution on the real line; in particular, this concept avoids the restriction that the distribution be absolutely continuous or purely discrete. These latter authors have also extended the concept of mean residual life function and have given related representations for distributions. In this paper, we introduce the concepts of hazard measure ordering and mean residual life ordering to compare two arbitrary probability distributions and study their basic properties.

Key words: Partial Ordering; Hazard Rate; Hazard Measure; Mean Residual Life Function; Stochastic Ordering;

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1 Introduction

Partial orderings relative to probability distributions is an important criterion in probability and statistics. There are several ways in which one can assert that a random variable X (or equivalently, the corresponding distribution function F) is grater than another random variable (or its distribution). The simplest way to compare two distribution functions is via their means (if they exist) or their variances (when the means are equal). However, such comparisons usually are not informative, because they are based on only one or two specific characteristics. In reliability theory, usually partial ordering of life distributions are based on functions such as survival function, hazard rate (HR) and mean residual life (MRL). There is an extensive literature dealing with the subject of partial orderings between distributions and their preservations under reliability operations such as convoluting, mixing and adopting $k-out-of-n$ systems. In this note, we study certain types of partial orderings between univariate distributions and their relationships, introducing, in the process of doing so, some new partial ordering in terms of the generalized concepts introduced by Kotz and Shanbhag (1980); most of the literature in reliability assumes that the distributions are absolutely continuous or purely discrete, and we try to escape here from such constraints. Kupka and Loo (1989) have previously introduced and studied some reliability properties along the lines of the present investigation.

2 Some basic definitions and auxiliary results

We need the following definitions and auxiliary results in the present investigation:

Definition 2.1 Let X be a real-valued random variable with $E(X^+) < \infty$. Define a real-valued Borel measurable function m on R satisfying

$$m(x) = E(X - x | X \geq x) \tag{1}$$

for all x such that $P(X \geq x) > 0$. This function is called *the mean residual life function* (MRL function for short).

Definition 2.2 Let F be a distribution function on R. Consider the measure ν_F on (the Borel $\sigma-field$ of) R such that

$$\nu_F(B) = \int_B \frac{dF(x)}{(1 - F(x-))} \tag{2}$$

for every Borel set B. This measure is called the hazard measure related to F.

Theorem 2.3 *Let $b(\leq \infty)$ denote the right extremity of the distribution function F of a random variable X with $E(X^+) < \infty$ and m be its MRL function. Further, $A = \{y : \lim_{x \uparrow y} m(x)$ exists and equal $0\}$. Then $b = \infty$ if A is empty and $b = inf\{y : y \in A\}$ if A is non-empty. Moreover, for every $-\infty < y < x < b$*

$$\frac{1 - F(x-)}{1 - F(y-)} = \frac{m(y)}{m(x)} exp\{-\int_y^x \frac{dz}{m(z)}\}, \tag{3}$$

and for every $-\infty < x < b$, $1 - F(x-)$ is given by the limit of the right hand side of (3) as $y \to -\infty$.

Corollary 2.4 *Let X be a non-negative continuous random variable with distribution function F and $E(X) < \infty$ and let b be the right extremity of F. Then, for every $x \in [0, b)$,*

$$1 - F(x-) = \frac{m(0)}{m(x)} exp\{-\int_0^x \frac{dz}{m(z)}\}, \tag{4}$$

where m is as defined in Theorem 2.3.

Theorem 2.5 *Let ν_F be as defined above and ν_F^c be continuous (non-atomic) part of ν_F and let $H_c(x) = \nu_F^c(-\infty, x]$. Denote by b the right extremity of F. Then $b = sup\{x : \nu_F(x, x+\delta) > 0\}$ for some $\delta > 0$, and the survival function $\bar{F}(x) = 1 - F(x-)$ is given by*

$$\bar{F}(x) = [\prod_{x_r \in D_x} (1 - \nu_F(x_r))] exp\{-H_c(x)\} \quad x < b, \tag{5}$$

where D_x is the set of all points $y \in (-\infty, x)$ such that $\nu_F\{y\} > 0$. ((2.5) also holds with '$x < b$' replaced by '$x \in R$' provided we define $exp\{-\infty\}$ to be equal to zero.)

Corollary 2.6 *If $-\infty < \alpha \leq \infty$ and the restriction of F to $(-\infty, \alpha)$ is continuous (i.e. if ν_F is continuous or non-atomic on $(-\infty, \alpha)$), then*

$$\bar{F}(x) = \exp\{-H(x)\} \quad \textit{for all } x \in (-\infty, \alpha),$$

where $H(x) = \nu_F((-\infty, x])$ and we define $\exp\{-\infty\} = 0$.

We have taken the definitions and results appearing above from Kotz and Shanbhag (1980). Specialized versions or variants of these have appeared in Cox (1962, 1972), Jacod (1975) and other places.

3 Main Results

Let X and Y be two real random variables with distribution functions F and G, respectively and let ν_F and ν_G be the hazard measures and m_F and m_G, the mean residual lives relative to F and G respectively. First we begin with the definitions of some existing partial orderings, with obvious modifications wherever appropriate in the light of the findings of Kotz and Shanbhag (1980).

Definition 3.1 The random variable X is said to be smaller than the random variable Y in the usual stochastic order, denoted by $X \leq_{st} Y$ ($Y \geq_{st} X$) if

$$\bar{F}(x) \leq \bar{G}(x), \qquad \textit{for all } x \in R, \tag{6}$$

where $\bar{F}(x) = 1 - F(x)$ and $\bar{G}(x) = 1 - G(x)$ are survival functions of X and Y, respectively.

The usual stochastic ordering is being used in many areas of statistics and applied probability. For some standard references on this, we refer the reader to Marshall and Olkin (1979), Ross (1983) and Shaked and Shanthikumar (1994).

Definition 3.2 Let X and Y be two random variables with X^+ and Y^+ integrable and MRL functions m_F and m_G respectively. Then X is said to be smaller than Y in the MRL order, denoted by $X \leq_{mr} Y$, if

$$m_F(x) \leq m_G(x) \quad for\ all\ x < min\{b_X, b_Y\}, \tag{7}$$

where b_X and b_Y are the right extremities of the distributions of X and Y respectively.

It can be easily shown that $X \leq_{mr} Y$ if and only if, in obvious notation,

$$\frac{\int_x^\infty \bar{F}(t)dt}{\int_x^\infty \bar{G}(t)dt}$$

is non-increasing on $\{x : \bar{G}(x) > 0\}$ and it implies that $b_X \leq b_Y$.

Definition 3.3 The random variable Y is said to be less than the random variable X in the the hazard measure order, denoted by $Y \leq_{hm} X$, if $\nu_G = \nu_F + \mu$ with μ as a non-negative measure on R, where ν_F and ν_G are the hazard measures of F and G, respectively.

In particular, if F and G are absolutely continuous (w.r.t. Lebesgue measure), we shall refer to hazard measure ordering as the hazard rate ordering and take 'hr' in place of 'hm'. It is well known that in the restrictive case referred to here, distribution functions F and G possess the 'hr' ordering if and only if $\frac{\bar{F}(x)}{\bar{G}(x)}$ is an increasing function on $\{x \in R : \bar{G}(x) > 0\}$ and $\nu_F(\{x \in R : \bar{G}(x) = 0\}) = 0$ We shall now show that the result in question holds without the absolute continuity of F and G.

Theorem 3.4 *Let X and Y be random variables distributed with df's F and G respectively. Then Y is less than X in the hazard measure order if and only if $\frac{\bar{F}(x)}{\bar{G}(x)}$ is an increasing function on $\{x \in R : \bar{G}(x) > 0\}$ and $\nu_F(\{x \in R : \bar{G}(x) = 0\}) = 0$*

Proof: We have from representation (5)

$$\begin{aligned} \frac{\bar{F}(x)}{\bar{G}(x)} &= \exp\{(\sum_{x_r \in D_x^{(F)} \cup D_x^{(G)}} (\log(1 - \nu_F\{x_r\}) - \log(1 - \nu_G\{x_r\}))) \\ &- (H_c^{(F)}(x) - H_c^{(G)}(x))\}, \quad x < min\{b_F, b_G\}, \end{aligned} \tag{8}$$

where b_F and b_G are the right extremities of F and G respectively, $D_x^{(F)}$ is the set of discontinuity points of ν_F lying in $(-\infty, x)$ and $H_c^{(F)}(x) = \nu_{F,c}((-\infty, x])$ with $\nu_{F,c}$ as the non-atomic part of ν_F (and $D_x^{(G)}$ and $H_c^{(G)}(x)$ are defined similarly for G). If $y \leq x < min\{b_F, b_G\}$ and $Y \leq_{hm} X$, then it follows from (8) that

$$\frac{\bar{F}(x)}{\bar{G}(x)} \geq \frac{\bar{F}(y)}{\bar{G}(y)}; \tag{9}$$

to see this note that

$$\begin{aligned}\frac{\bar F(x)}{\bar G(x)} &= \frac{\bar F(y)}{\bar G(y)}\exp\{\sum_{x_r\in(D_x^{(F)}\cup D_x^{(G)})\setminus(D_y^{(F)}\cup D_y^{(G)})}\sum_{k=1}^{\infty}\{(\nu_G\{x_r\})^k-(\nu_F\{x_r\})^k\}/k\\ &+ \nu_{G,c}((y,x])-\nu_{F,c}((y,x])\}.\end{aligned}$$

The assertion $Y \leq_{hm} X$ implies that $b_F = b_G$. (Note that $b_F > b_G \Longrightarrow \nu_F((b_G,\infty)) > \nu_G((b_G,\infty))$, and $b_F < b_G \Longrightarrow$ for an a smaller than b_F and sufficiently close to b_F, $\nu_F((a,b_F]) > \nu_G((a,b_F])$.) Because of the left continuity of $\bar F$ and $\bar G$ we have then the 'only if' part of the result on view of (9). To prove the '*if*' part, note that under the given condition, $b_F = b_G$ and the exponent of (8) is an increasing left continuous function on $(-\infty, b_F)$ with limit as $x \longrightarrow -\infty$ to be zero. Consequently it follows that for each $x \in (-\infty, b_F)$, $\nu_F\{x\} \leq \nu_G\{x\}$, and we have $\nu_{G,c}(.\cap(-\infty,b_F)) = \nu_{F,c}(.\cap(-\infty,b_F)) + \mu(.)$ with μ as a non-negative measure. As $\nu_F(\{z\in R : \bar G(z) = 0\}) = 0$, we have then that $Y \leq_{hm} X$. { Note that if b_F is a discontinuity point of F i.e $b_F < \infty$ with $\nu_F\{b_F\} = 1$, it is also a discontinuity point of G giving $\nu_G\{b_G\} = 1$.}
Hence we have the result.

The following theorem and example show that the hazard measure ordering is stronger than the usual stochastic ordering.

Theorem 3.5 *If X_1 and X_2 are two random variables such that $X_2 \leq_{hm} X_1$, then $X_2 \leq_{st} X_1$.*

Proof: Let $\bar F_1$ and $\bar F_2$ denote the survival functions and ν_1 and ν_2 denote the hazard measures of X_1 and X_2 respectively. Then by the representation 5 we have the right extremities of F_1, the distribution function of X_1, and F_2, the distribution function of X_2 to be equal with

$$\bar F_i(x) = [\prod_{x_r\in D_{ix}}(1-\nu_i\{x_r\})]exp\{-H_{ic}(x)\} \quad x\in(-\infty,b), \tag{10}$$

$i = 1,2$, where b is the common right extremity of F_1 and F_2, D_{ix} is the set of all points $y \in (-\infty, x)$ such that $\nu_i\{y\} > 0$ and $H_{ic}(x) = \nu_i^c(-\infty, x]$ where ν_i^c is the continuous part of ν_i, where ν_i is the hazard measure relative to F_i. Now, since $\nu_2(D) \geq \nu_1(D)$ for every Borel set D, we can easily seen that $\bar F_2(x) \leq \bar F_1(x)$ for every $x \in R$.

Example 3.6 Let X_2 be a non-degenerate random variable with survival function $\bar F_2$ and right extremity $b < \infty$. Define $X_1 = X_2 - 1$; the survival function of X_1, $\bar F_1$, is given by

$$\bar F_1(x) = \bar F_2(x+1), \quad x \in R.$$

Clearly then $X_2 \leq_{st} X_1$ but the right extremities of X_1 and X_2 are different and hence they do not obey the corresponding hazard measure ordering. There exist also other examples in the literature (see for example Alzaid (1988) and Shaked and Shanthikumar (1991).

The MRL ordering is a well known concept in the literature, where it is assumed usually that random variables are non-negative and absolutely continuous. It is known that for absolutely continuous random variables, the MRL ordering is weaker than hr ordering (see Shaked and Shanthikumar (1994)). Indeed this last result with 'hr' replaced by 'hm' holds for more general random variables that are not necessarily absolutely continuous and non-negative.

Theorem 3.7 *If $Y \leq_{hm} X$, and X^+ and Y^+ are integrable then $Y \leq_{mr} X$.*

Proof: Suppose $Y \leq_{hm} X$. As observed in the proof of Theorem 3.4, this implies that $b_G = b_F$, where b_F and b_G are the right extremities of the distributions of X and Y respectively. In view of Theorem 3.4, with the notation used in the theorem, we have then

$$\frac{\bar{F}(x)}{\bar{G}(x)} \leq \frac{\bar{F}(x+t)}{\bar{G}(x+t)}, , \quad x, x+t \in (-\infty, b_F) \text{ and } t > 0, \tag{11}$$

this gives

$$\frac{\int_0^{b_F - x} \bar{G}(x+t)dt}{\bar{G}(x)} \leq \frac{\int_0^{b_F - x} \bar{F}(x+t)dt}{\bar{F}(x)}, \quad x \in (-\infty, b_F), \tag{12}$$

yielding that $Y \leq_{mr} X$.

Singh and Vijayaree (1991), using a counter example, showed that the MRL ordering is not closed under the formation of $k - out - of - n$ systems. A comparison of random sums based on the MRL ordering is studied by Pellery (1993), while Shaked and Shanthikumar (1994) showed that under some conditions the MRL ordering is preserved under the operation of taking convolution. Shaked and Shanthikumar (1991) proved that under the condition that the ratio of the MRL's of X and Y is increasing, the HR ordering and the MRL ordering are equivalent.

The following theorem now shows that the Shaked-Shanthikumar (1991) findings remain valid even when the assumption that random variables are non-negative and absolutely continuous is dropped.

Theorem 3.8 *Let X_i, $i = 1, 2$ be two random variables with X_i^+, $i = 1, 2$ integrable and mean residual life functions m_i, $i = 1, 2$, respectively. Let b_{X_i}, $i = 1, 2$ be the right extremities of the distribution of X_i, $i = 1, 2$ respectively. Suppose that $\frac{m_1(x)}{m_2(x)}$ increases for $x < min\{b_{X_1}, b_{X_2}\}$. Then $X_1 \leq_{mr} X_2$ implies $X_1 \leq_{hm} X_2$.*

Proof: The assertion of the theorem can be proved as follows. As observed earlier, $X_1 \leq_{mr} X_2$ implies that the right extremity of the F_1, the distribution of X_1, is less than or equal to that of F_2, the distribution of X_2. The increasing nature of $\frac{m_1(x)}{m_2(x)}$ for $x < min\{b_{X_1}, b_{X_2}\}$ implies that $b_{X_1} \geq b_{X_2}$ and hence we have that $b_{X_1} = b_{X_2}$. On the other hand, we have

$$\frac{\bar{F}_2(x)}{\bar{F}_1(x)} = \frac{m_1(x)}{m_2(x)} \cdot \frac{\int_x \bar{F}_2(x)dx}{\int_x \bar{F}_1(x)dx}, \quad x < min\{b_{X_1}, b_{X_2}\}.$$

Under the assumptions of the theorem, in view of what we have observed immediately after Definition 3.2 and the fact that $b_{X_1} = b_{X_2}$, we have the right hand side of the last equality, and hence its left hand side, to be increasing on $\{x \in R : \bar{F}_1(x) > 0\}$ and $\nu_{F_2}(\{x \in R : \bar{F}_1(x) = 0\}) = 0$. By Theorem 3.4, we have then $X_1 \leq_{hm} X_2$, and the theorem is proved.

Theorem 3.9 *Let $Y \leq_{hm} X$ and Z be a continuous random variable independent of X and Y such that $P\{X \geq Z\} > 0$ (and hence also such that) $P\{Y \geq Z\} > 0$). Then*

$$(X|X \geq Z) \geq_{st} (Y|Y \geq Z) \tag{13}$$

Proof: Let us denote by F, G and H respectively the *df's* of X, Y and Z. We can then see that (13) is equivalent to

$$\int_{[x,\infty)} H(y)dF(y) \int_R H(y)dG(y) - \int_{[x,\infty)} H(y)dG(y) \int_R H(y)dF(y) \geq 0 \text{ for all } x \in R. \tag{14}$$

We can see that (13) is equivalent to the condition that

$$\int_{[x,\infty)} H(y)dF(y) \int_{(-\infty,x]} H(y)dG(y) - \int_{[x,\infty)} H(y)dG(y) \int_{(-\infty,x]} H(y)dF(y) \geq 0 \quad \text{for all } x \in R, \tag{15}$$

which, in turn, is then seen to be equivalent, in view of Fubini's theorem, to

$$\begin{aligned} &\int_R min\{\bar{F}(x), \bar{F}(z)\}dH(z) \int_{(-\infty,x]} (\bar{G}(z) - \bar{G}(x))dH(z) \\ &- \int_R min\{\bar{G}(x), \bar{G}(z)\}dH(z) \int_{(-\infty,x]} (\bar{F}(z) - \bar{F}(x))dH(z) \geq 0 \text{ for all } x \in R. \end{aligned} \tag{16}$$

As the inequality in (16) is met trivially when $\bar{G}(x) = 0$, it is clear that to have (13), it is sufficient if we show that for each x with $\bar{G}(x) > 0$ and hence $\bar{F}(x) > 0$,

$$\begin{aligned} &\bar{F}(x)\bar{G}(x)\{\int_R min\{1, \frac{\bar{F}(z)}{\bar{F}(x)}\}dH(z) \int_{(-\infty,x]} (\frac{\bar{G}(z)}{\bar{G}(x)} - 1)dH(z) \\ &- \int_R min\{1, \frac{\bar{G}(z)}{\bar{G}(x)}\}dH(z) \int_{(-\infty,x]} (\frac{\bar{F}(z)}{\bar{F}(x)} - 1)dH(z)\} \geq 0, \quad \text{for all } x \in R, \end{aligned} \tag{17}$$

where we read $\frac{\bar{F}(z)}{\bar{F}(x)} = 0$ if $\bar{F}(x) = 0$ and $\frac{\bar{G}(z)}{\bar{G}(x)} = 0$ if $\bar{G}(x) = 0$. In view of Theorem 3.4, we have then the theorem.

Corollary 3.10 *Let $Y \leq_{hm} X$ and Z be a continuous random variable independent of X and Y. Then, for all z with $P\{X + Z \geq z\} > 0$ (and hence $P\{Y + Z \geq z\} > 0$), we have*

$$(X|X + Z \geq z) \geq_{st} (Y|Y + Z \geq z). \tag{18}$$

Proof: The result follows on applying the theorem with $z - Z$ in place of Z with z arbitrary.

Corollary 3.11 *If the assumptions of Corollary 3.10 are met with distribution of Z as absolutely continuous with increasing hazard function, on its support, when the support id assumed to be an interval, then*

$$Y + Z \leq_{hm} X + Z. \tag{19}$$

Proof: The result follows from Corollary 3.10 as (19) is equivalent to that

$$\frac{\int_R h(z-x)dF(x)}{\int_R \bar{H}(z-x)dF(x)} \leq \frac{\int_R h(z-x)dG(x)}{\int_R \bar{H}(z-x)dG(x)}, \quad z < b_Z + b_X, \tag{20}$$

where b_X and b_Z are the right extremities of the distribution of X and Z respectively, h is the density corresponding to Z and $\bar{H}$ is the survival function relative to Z. (Note that $\bar{H}(z-.)$ is the *df* of $z - Z$ and we have, in obvious notation, $b_X = b_Z$.)

Remark 3.12 Given any finite measure μ on (the Borel σ-field of) R there exists a sequence $\{\mu_n : n = 1, 2, ...\}$ of measures on R, that are absolutely continuous with respect to Lebesgue measure, converging weakly to μ. In view of this. Theorem 3.9 implies that given $Y \leq_{hm} X$, we have sequences $\{Y_n : n = 1, 2, ...\}$ and $\{X_n : n = 1, 2, ...\}$ of random variables with distributions that are absolutely continuous with respect to Lebesgue measure, so that $Y_n \leq_{hm} X_n$ for each n. (Note that G can be expressed as product of $\bar{F}$ and the survival function of a finite measure on R.) Also, if Z is independent of X and Y, we can claim the existence of a sequence $\{Z_n : n = 1, 2, ...\}$ of random variables with distributions that are absolutely continuous with respect to Lebesgue measure, so that for each n, Z_n is independent of X_n and Y_n and the sequence $\{Z_n : n = 1, 2, ...\}$ converges in distribution to Z. In view of observations, it is clear that Theorem 3.9, Corollary 3.10 and Corollary 3.11 follow also from the corresponding results when X, Y and Z have distributions that are absolutely continuous with respect to Lebesgue measure. (To have, in particular, Corollary 3.11 here, appeal to the stability theorem relative to hazard measures, given by Kotz and Shanbhag (1980).)

Remark 3.13 The definition of $Y \leq_{hm} X$ above is tailored so as to subsume the definition in Shaked and Shantikumar (1994) of $Y \leq_{hr} X$ as a special case. However, the definition given in Shaked and Shantikumar (1994) is not universally followed; indeed, there are places in which the ordering "$Y \leq_{hr} X$" under the stated conditions is referred to as "$X \leq_{hr} Y$. With obvious alternations in the notation used in our results, one can produce the relevant results that generalize the results in the literature employing the latter notation.

Appealing to Corollary 3.11, we can easily get the following results as further corollaries of the theorem.

Theorem 3.14 *Let X and Y be two independent random variables with distributions that are absolutely continuous (with respect to Lebesgue measure) with supports as intervals and increasing hazard rates on the respective supports. Then $X + Y$ has an absolutely continuous distribution with interval support and increasing hazard rate on the support.*

Proof: A random variable Z with absolutely continuous distribution having infinite right right extremity and continuous density has an increasing hazard rate if and only if $Z \leq_{hm} Z + t$ for each $t > 0$; the "only if" part of the assertion holds even when the assumption that the density is continuous is not met. We can construct a sequence $\{X_n : n = 1, 2, ...\}$ of random variables converging in distribution to X, independent of a random variable distributed as Y, such that $X_n^{'}$s have increasing hazard rate absolutely continuous distributions with infinite right extremities. (For example, if $\bar{F}$ is the survivor function of X and, for each $n \geq 1$, x_n is a point such that $\bar{F}(x_n) = \frac{1}{n+1}$ and the hazard rate at x_n is y_n, then one can take X_n such that

$$P\{X_n \geq x\} = \begin{cases} \bar{F}(x) & \text{if } x \leq x_n \\ \bar{F}(x_n)e^{-y_n(x-x_n)} & \text{if } x \geq x_n.) \end{cases}$$

We have then $X_n \leq X_n + t$ for each $t > 0$ and $n = 1, 2, ...$ Suppose Y^* is the random variable distributed as Y and independent of $\{X_n\}$. As Y^* has an absolutely continuous distribution with interval support and increasing hazard rate on the support, Corollary 3.11 implies that $X_n + Y^* \leq_{hm} X_n + Y^* + t$ for each $t > 0$ and $n = 1, 2, ...$ As $\{X_n^* + Y_n^*\}$ converges in distribution to $X + Y$ and $X + Y$ has an absolutely continuous distribution, the stability theorem for hazard measures given, for example, in Kotz and Shanbhag (1980) implies then that

$$X + Y \leq_{hm} X + Y + t \quad \text{for all } t > 0. \tag{21}$$

It is obvious that the distribution of $X + Y$ has its support to be an interval and density (i.e. some version of it) to be continuous. In view of this, (21) implies that the theorem holds.

Corollary 3.15 *Let (X_i, Y_i), $i = 1, 2, ..., m$ be independent random vectors such that $Y_i \leq_{hm} X_i$, $i = 1, 2, ..., m$. If X_i and Y_i, $i = 1, 2, ..., m$ have absolutely continuous distributions with interval supports and increasing hazard rates on respective supports, then*

$$\sum_{i=1}^{m} Y_i \leq_{hm} \sum_{i=1}^{m} X_i;$$

(Also Theorem 3.14 implies that the distribution of $\sum_{i=1}^{m} X_i$ and $\sum_{i=1}^{m} Y_i$ have interval supports with increasing hazard rates on respective supports.)

Proof: We shall obtain the result by induction. Assume that it is valid when $m = k$, where k is a fixed positive integer. Then, if we define (X_{k+1}, Y_{k+1}) to be a random vector independent of (X_i, Y_i), $i = 1, 2, ..., k$ and distributed as (X_1, Y_1), we have by Corollary 3.11 and Theorem 3.14 respectively

$$
\begin{aligned}
(\sum_{i=1}^{k} Y_i) + Y_{k+1} \quad &\leq_{hm} \quad (\sum_{i=1}^{k} X_i) + Y_{k+1} \\
&\leq_{hm} \quad (\sum_{i=1}^{k} X_i) + X_{k+1}
\end{aligned} \tag{22}
$$

(on noting that Theorem 3.14 implies that $\sum_{i=1}^{k} X_i$ has an increasing hazard rate absolutely continuous distribution). Hence we have that the result holds for $m = k+1$. As the result trivially holds for $m = 1$, it follows then inductively that the result holds for all m.

Remark 3.16 In 1.B.1 on page 12 of Shaked and Shanthikumar (1994), a misleading statement has appeared. Without clarifying what really is meant by the hazard rate corresponding to a distribution that is not absolutely continuous with respect to Lebesgue measure, the authors claim that their definition of hazard rate ordering, possibly with a modification, holds even when the distributions are not assumed to be absolutely continuous. Our findings in this paper provide one with a clear picture of the situation in this

Remark 3.17 For an absolutely continuous distribution function F with density function f, the reverse hazard rate is defined by $\frac{f(x)}{F(x)}$ on $\{x : F(x) > 0\}$. Taking a hint from this, if G is a *df* on R, we can define the reverse hazard measure relative to G as the measure ν_G' on R such that for every Borel set B

$$\nu_G' = \int_B \frac{1}{F(x)} dP_F(x),$$

where P_F is the measure determined by F, on R. Note that for every Borel set B,

$$\nu_G'(B) = \nu_H(-B),$$

where H is *df* given by

$$H(x) = 1 - G(-x), \quad x \in R,$$

and ν_H is the hazard measure relative to H. Implications of this to our study are self-evident.

Theorem 3.18 *Let F be a probability distribution function on R and $\bar{F}$ be the corresponding survival function, and, for each $\alpha > 0$, let X_α and Y_α be random variables with survival functions $\bar{F}^\alpha(x)$, $x \in R$ and $1 - F^\alpha(x)$, $x \in R$, respectively. Then, with obvious terminology, we have $\{X_\alpha\ \alpha > 0\}$ decreasing in hazard measure and $\{Y_\alpha; \alpha > 0\}$ increasing in hazard measure.*

Proof: If $0 < \alpha < \infty$, then for any Borel subset B of the set of points at which F is continuous, we have the value of the hazard measure relative to the survival function $\bar{F}^\alpha$ to be $\int_B \frac{\alpha}{\bar{F}(x)} dF(x)$, and, for any $\{x\}$ with $x \in R$ or in particular, as a discontinuity of F, we have the corresponding value (in obvious notation) of the hazard measure to be $1-(1-\nu_F\{x\})^\alpha$. Hence the first part of the assertion follows easily. To prove the second part of the assertion, note that if α and B are as above, then the value for B of the hazard measure relative to the survival function $1-F^\alpha(x)$, $x \in R$ equals $\int_B \frac{\alpha F^{\alpha-1}(x)}{1-F^\alpha(x)} dF(x)$, which is decreasing in α if α is allowed to vary (because, for each $y \in (0,1)$, $\frac{\alpha y^{\alpha-1}}{1-y^\alpha} = (\int_y^1 (z/y)^{\alpha-1} dz)^{-1}$). Moreover, in this latter case, for any $\{x\}$ with x as a discontinuity point of F, we have the value of the hazard measure as

$$\frac{F^\alpha(x)-F^\alpha(x-)}{1-F^\alpha(x-)} = \left(1+\frac{1-F^\alpha(x)}{F^\alpha(x)-F^\alpha(x-)}\right)^{-1},$$

which is decreasing in α when α is allowed vary (because if $0<y<z<1$, we have $\frac{1-z^\alpha}{z^\alpha-y^\alpha} = \int_z^1 \{\int_y^z (\frac{u}{v})^{\alpha-1} du\}^{-1} dv$).

In view of what we have observed, we have then the second part of the assertion and hence the theorem.

Corollary 3.19 *Let $\{X_n : n = 1, 2, ...\}$ be a sequence of independent identically distributed random variables. Then, for each integer $m \geq 2$,*

$$min\{X_1, ..., X_m\} \leq_{hm} min\{X_1, ..., X_{m-1}\}$$

and

$$max\{X_1, ..., X_{m-1}\} \leq_{hm} max\{X_1, ..., X_m\}$$

Proof: If we denote the distribution function and survival function of X_1 by F and $\bar{F}$ respectively, then for each positive integer n, we have the survival functions of $\min\{X_1, ..., X_n\}$ and $\max\{X_1, ..., X_n\}$ to be respectively $\bar{F}^n(x)$, $x \in R$, and $1-F^n(x)$, $x \in R$. On appealing to the theorem, we have the corollary.

Remark 3.20 Theorems 1.B.15 and 1.B.16 in Shaked and Shanthikumar (1994) are not valid in their existing forms. (We do not claim here that typos or minor blemishes have noting to do with this.) The following example illustrates as why this so

Example 3.21 Let X_1 and X_2 be independent random variables with absolutely continuous (with respect to Lebesgue measure) distribution functions F_1 and F_2 respectively with supports $[0, a]$ and $[a, \infty)$, where $a \in (0, \infty)$. Note that we have here $\min\{X_1, X_2\} = X_1$ almost surely and $\max\{X_1, X_2\} = X_2$ almost surely, and it is not true that $X_1 \leq_{hr} X_2$, which contradicts Theorem 1.B.15 of the cited reference (claiming that, in the present case, $\min\{X_1, X_2\} \leq_{hr} \max\{X_1, X_2\}$). Also, on taking, Y_1 and Y_2 to be independent random variables with distribution functions $\alpha F_1 + (1-\alpha)F_2$ and F_2 with $\alpha \in (0,1)$ and F_1 and F_2 as defined above, respectively, we have that the hazard rate of Y_2 is greater than or equal to that of Y_1 almost everywhere. However, it is not true that in this latter case, we have $Y_1 \leq_{hr} \min\{Y_1, Y_2\}$, contrdicting Theorem 1.B.16 of the cited reference.

Remark 3.22 Let (X, Z_1) and (Y, Z_2) be independent random vectors such that

$$X|Z_1 \leq_{hm} Y|Z_2 \qquad \text{almost surely.}$$

Then we have $X \leq_{hm} Y$. (This follows on noting that

$$P\{X \geq x\}P\{Y \geq y\} \leq P\{X \geq y\}P\{Y \geq x\} \quad \text{if } x \leq y,$$

because

$$P\{X \geq x|Z_1\}P\{Y \geq y|Z_2\} \leq P\{X \geq y|Z_1\}P\{Y \geq x|Z_2\} \quad a.s.,\ x \leq y,)$$

and that X and Y have the same right extremity. (This result essentially extends Theorem 1.B.8 of Shaked and Shanthikumar (1994).)

Remark 3.23 Using a straight forward argument, essentially a minor version of that given by Shaked and Shanthikumar (1994) to prove their Theorem 1.B.4., one can stablish that the following generalized version of the theorem referred to of Shaked and Shanthikumar holds:

Theorem 3.24 *Let (X_i, Y_i), $i = 1, 2, ..., m$ be independent random vectors such that $X_i \leq Y_i$, $i = 1, 2, ..., m$. Suppose that X_i's are i.i.d. and so also are Y_i's and Y_i's are continuous. Then, in standard notation of order statistics*

$$X_{(k)} \leq_{hm} Y_{(k)}, \quad k = 1, 2, ..., m.$$

(The question as to whether the theorem holds when the assumption that Y is continuous is dropped, remains open.)

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